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VARIATIONAL METHOD OF DETERMINING THE HYDRODYNAMIC PARAMETERS IN CONVECTIVE HEAT-TRANSFER PROBLEMS FOR SEPARATION FLOWS IN CHANNELS

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UDC 532.542:536.242

A method is given for computing viscous fluid flows in a channel by using a variational formulation.

The application of iteration methods [1-6] to compute the convective heat transfer in a separation flow in a channel requires substantial expenditures of machine time, which is associated mainly with the slow convergence of the iteration process for the hydrodynamic equations.

Utilization of direct methods, including variational, results in a significant reduction in the computation time, as a rule, although it also complicates the algorithm for the solution.

A certain hybrid scheme is proposed in this paper that combines finite-difference and variational-difference computation schemes, which turn out to be relatively simply in realization on an electronic computer while at the same time sufficiently economical in the sense of the computation time.

The scheme is developed in application to specific cases of the flow in cylindrical or plane channels behind a sudden expansion and is based on an explicit method for solving all equation in the longitudinal (cruising) coordinate x and an implicit method in the transverse coordinate y.

A feature of the method is that the solution is sought in the form $u = \overline{u} + \delta u$; $v = \overline{v} + \delta v$, where $\overline{u}$ is the first approximation obtained from (1) by the factorization method, $\overline{v}$ is determined from the continuity equation (3), δu, δv are the refining corrections obtained from the condition of minimum work of the hydrodynamic forces on a finite set of closed contours (closedness of the contour permits elimination of the pressure from a number of unknowns).

The method mentioned permits obtaining a "good" solution more rapidly for the problem under consideration than in [1-5], say, because of the abrupt reduction in the number of iterations typical for variational methods. At the same time, such a combined approach is simpler, and (in this case) more economical than the ap-

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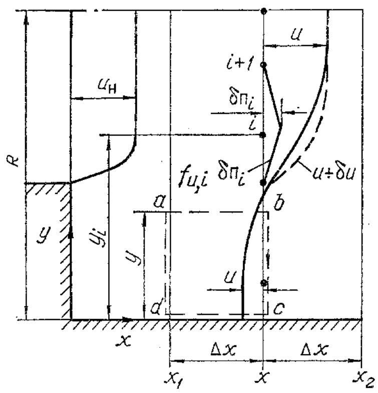

Fig. 1. Computational scheme (the dashes denote the contour of integration).

plication of the method of finite elements in "pure form" since it permits reduction of the number of unknown parameters (in the particular case elements) to a minimum. This latter becomes possible because the shape of the velocity profile in the elements is determined first by a finite-difference method.

Let us examine the crux of the method in an example of stationary flow in a symmetric plane channel behind an obstacle (for the case with constant density and variable viscosity).

We represent the equations of motion and continuity in the form

$$F_x + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{1}$$

$$F_y + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0, \tag{2}$$

$$(\partial u/\partial x) + (\partial v/\partial y) = 0, \tag{3}$$

where

$$F_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{xy}}{\partial y};$$

$$F_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \tau_{yy}}{\partial y};$$

$$\tau_{xx} = 2\mu/\rho\,(\partial u/\partial x);\ \ \tau_{yy} = 2\mu/\rho\,(\partial v/\partial y);$$

$$\tau_{xy} = \mu/\rho\,(\partial u/\partial y + \partial v/\partial x);\ \ \mu = \mu_т + \mu_л.$$

Let us assume the following boundary conditions

$$x = 0\ \ p = p_н(y);\ \ u = u_н(y);\ \ v = 0;\ \ y = 0\ \ u = v = 0; \tag{4}$$
$$y = R\ \ (\partial u/\partial y)_R = 0;\ \ v_R = 0;$$

at the output $\partial u/\partial x = \partial v/\partial x = 0$.

In the first iteration the following initial conditions are satisfied

$$u(x + \Delta x,\ y) = u(x - \Delta x,\ y);\ \ v(x + \Delta x,\ y) = v(x - \Delta x,\ y)$$

(successively, as going from one section x to another).

Let us now calculate the work Φ of the hydrodynamic forces $\overline{F}$ (inertia, viscosity, and pressure) over the closed contour a, b, c, d (dashes in Fig. 1, clockwise integration):

$$\Phi = \oint_{abcd} \overline{F}d\overline{l} = \int_{ab} F_x dx + (p_b - p_a)/\rho + \int_{bc} F_y dy + (p_c - p_b)/\rho +$$
$$+ \int_{cd} F_x dx + (p_d - p_c)/\rho + \int_{da} F_y dy + (p_a - p_d)/\rho. \tag{5}$$

Assuming the coefficients u and v of the convective terms to be constants on the section $ab = \Delta x$, and taking account of (4), we obtain (after cutting out the terms containing the pressure)

$$\Phi = u(u-u_s)z_u + v(\partial u/\partial y)\Delta x - \frac{z_u}{\Delta x}\int_0^y u(v-v_s)\,dy - 0.5v^2 - \tau_{xx} + \tau_{xx,1} - (\partial\tau_{xy}/\partial y)\Delta x + (\partial\tau_{xy}/\partial y)_0\Delta x +$$
$$+\frac{1}{\Delta x}\int_0^y (\tau_{xy}-\tau_{xy,1})\,dy + \tau_{yy} + \int_{da} F_y dy, \tag{6}$$

where the last integral is known from computations in the preceding step, the subscript s denotes utilization of upstream differences in the convective terms (in the coordinate x), $z_u = \operatorname{sign} u$, $u_s = u(x - \Delta x z_u)$ (we use central differences in the coordinate y).

Here and henceforth, the quantities with subscript 1, 2, 0 refer to the sections $x = x_1$, $x = x_2$, $y = 0$, respectively, and rest to the section $x = x$.

Now, let us determine the functional of x minimizable in this section:

$$J = \int_0^R \Phi^2 dy. \tag{7}$$

The expression (7) is the sum of squares of the work of the hydrodynamic forces on the set of rectangular contours (of the type a, b, c, d) with dimensions Δx, y. Evidently $J \geq 0$. We now find a solution for u and v such that $J \to \min$ under the additional condition $v_R = 0$. To do this, we partition the transverse section of the channel into one-dimensional elements delineated by the control nodes with the numbers i = 1, 2, ..., I and we represent the desired solution in the form

$$u = \bar{u} + \delta u = \bar{u} + \sum_{i=1}^{I} f_{u,i}\Pi_i,$$
$$v = \bar{v} + \delta v = \bar{v} + \sum_{i=1}^{I} f_{v,i}\Pi_i, \tag{8}$$

where $f_{u,i} \equiv f_{u,i}(y)$, $f_{v,i} \equiv f_{v,i}(y)$ are certain basis functions, $\bar{u}$ is the initial approximation obtained by solving (1) by the factorization method, and Π_i is a set of small parameters.

Substituting (8) into (6) and expanding the function obtained $\Phi = \Phi(\Pi_i, y)$ in a Taylor series and keeping second-order terms in the unknown small parameters Π_i in the neighborhood of the point $\Pi_i = 0$, we represent (7) in the form

$$J = \int_0^R \left[\Phi_0 + \sum_{i=1}^{I}\left(\frac{\partial\Phi}{\partial\Pi_i}\right)_0 \Pi_i + 0.5\sum_{i,j=1}^{I}\left(\frac{\partial^2\Phi}{\partial\Pi_i\partial\Pi_j}\right)_0 \Pi_i\Pi_j\right]^2 dy. \tag{9}$$

Differentiating (9) with respect to the parameters Π_i and taking into account the additional condition $v_R = 0$, we obtain the minimum condition in the form of a system of I + 1 linear equations (second- and higher-order terms are discarded)

$$B_{i,j}\Pi_i + \Lambda(\partial v_R/\partial\Pi_i) = C_i,\quad i = 1,\ 2\ldots I,$$
$$\sum_{i=1}^{I}(\partial v_R/\partial\Pi_i)\Pi_i + v_R = 0, \tag{10}$$

where Λ is the Lagrange parameter (among the unknowns):

$$B_{i,j} = \int_0^R \left[\left(\frac{\partial\Phi}{\partial\Pi_i}\right)_0 + \Phi_0\left(\frac{\partial^2\Phi}{\partial\Pi_i\partial\Pi_j}\right)_0\right] dy;$$
$$C_i = -\int_0^R \Phi_0(\partial\Phi/\partial\Pi_i)\,dy. \tag{11}$$

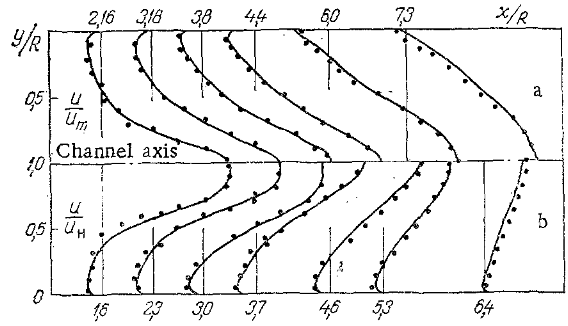

Fig. 2. Velocity profiles in a circular tube at different distances from the input (x/R): a) tests [7], d/D = 0.215; b) tests [8], d/D = 0.32.

The derivatives $(\partial\Phi/\partial\Pi_i)$ are evaluated by differentiating (6) and (8):

$$\frac{\partial\Phi}{\partial\Pi_i}=\frac{\partial\Phi}{\partial u}\frac{\partial u}{\partial\Pi_i}+\frac{\partial\Phi}{\partial v}\frac{\partial v}{\partial\Pi_i}.$$

As an illustration, let us present the derivatives of certain components on the right side of (6). Using the notation

$$\left.\begin{aligned}
&\Phi_1\equiv u(u-u_s);\ \Phi_2\equiv v(\partial u/\partial y);\ \Phi_3\equiv\int_0^y u(v-v_s)\,dy;\\
&\Phi_7\equiv\partial/\partial y\,[\mu(\partial u/\partial y+\partial v/\partial x)],
\end{aligned}\right.$$

we obtain

$$\left.\begin{aligned}
&\partial\Phi_1/\partial\Pi_i=f_{u,i}(2u-u_s);\ \partial^2\Phi_1/\partial\Pi_i\partial\Pi_j=2f_{u,i}f_{u,j};\\
&\partial\Phi_2/\partial\Pi_i=f_{v,i}(\partial u/\partial y)+f'_{u,i}v;\ f'_{u,i}=\partial f_{u,i}/\partial y;\\
&\partial^2\Phi_2/\partial\Pi_i\partial\Pi_j=f_{v,i}f'_{u,j}+f_{v,j}f'_{u,i};\\
&\partial\Phi_3/\partial\Pi_i=\int_0^y(f_{u,i}(v-v_1)+uf_{v,i})\,dy;\\
&\partial^2\Phi_3/\partial\Pi_i\partial\Pi_j=\int_0^y(f_{u,i}f_{v,j}+f_{u,j}f_{v,i})\,dy;\\
&\partial\Phi_7/\partial\Pi_i=\partial/\partial y\,(\mu f'_{u,i}+\mu f_{v,i}/\Delta x).
\end{aligned}\right\}\qquad(12)$$

We perform computations by means of (12) by setting $u=\bar{u}$, $v=\bar{v}$. We select piecewise continuous functions as basis functions for specific computations, and values of the velocity u at the control nodes as parameters

$$f_{u,i}(y)=1-\bar{y},$$

where i is the number of the control node

$$\begin{aligned}
&\bar{y}=(y-y_i)/(y_{i+1}-y_i),\ y_i\leqslant y\leqslant y_{i+1};\\
&\bar{y}=(y-y_i)/(y_{i-1}-y_i),\ y_{i-1}\leqslant y\leqslant y_i;\\
&\bar{y}=1,\ y\leqslant y_{i-1},\ y\geqslant y_{i+1}.
\end{aligned}\qquad(13)$$

Necessary to the conservation of (3) is

$$\bar{v}=-\frac{1}{\Delta x}\int_0^y(\bar{u}-u_1)\,dy;\ f_{v,i}=-\frac{1}{\Delta x}\int_0^y f_{u,i}dy.\qquad(14)$$

Determining the coefficients of the matrices B and C from (11)-(12) by using numerical integration and differentiation, we solve the system (10), we calculate the velocity increment in a given section

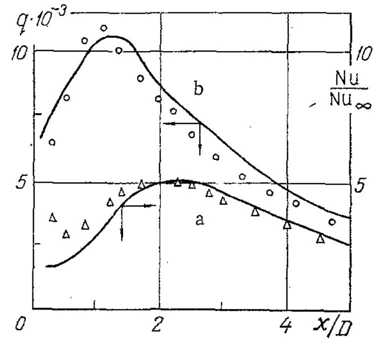

Fig. 3. Convective heat elimination in a circular tube at different distances from the input (x/D): a) tests [8]; d/D = 0.32; b) tests [9]; d/D = 0.25; q is the heat flux in W/m^2.

$$\delta u = \sum_{i=1}^{I} f_{u,i}\Pi_i, \quad \delta v = \sum_{i=1}^{I} f_{v,i}\Pi_i$$

and the pressures from (1) and (2). Then the computation is performed for the next section x. Three to five iterations in each step in x are required to achieve the minimum of the functional J in the initial sections of the channel ($x/D \le 0.5$), then their number is cut down to 1-2.

The finite-difference equations for the other variables, the temperature, kinetic energy of the turbulent pulsations, and the dissociation energy of these pulsations were solved by the factorization method by using the constants and boundary conditions for ($k-\varepsilon$) turbulence model from [2]. In the initial sections k and ε were given by the formulas

$$k = 1.5\,(\mathrm{Tu})^2 u_{\text{н},m}^2, \quad \varepsilon = 0.09\,k^{1.5}/(r l_{\text{т}}/R),$$

where l_T/R can be treated as a certain relative initial scale of turbulence.

For Tu = 0.01-0.05 and $l_T/R \le 0.03$ the computed curves satisfactorily describe the experimental data [7, 8] in both the hydrodynamics (Figs. 2a and b) and the heat transfer (Fig. 3a) obtained in regimes with low initial condition of turbulence. It should be noted that if the scale of turbulence l_T/R is taken less than 0.02, then the change in Tu from zero to one has practically no effect on the computation results. If the scale of turbulence is increased to 0.1, then the growth of Tu from zero to 0.2 will result in a significant reduction in the length of the recirculation zone, will shift the zone of maximal heat fluxes closer to the entrance, and increase their magnitude sharply (by 20-30%); under these initial conditions a definite correspondence with tests [9] conducted with a high initial level of turbulence is achieved (see Fig. 3b, where primary test data of [9] are presented; the air temperature at the entrance and channel walls are 900 and 300°K, respectively; the velocity at the entrance is 17.5 m/sec, d/D = 0.25, D = 1 m).

The results presented above were obtained for five iterations in the cruise coordinate x; in this case the computation time on an ES-1033 computer was 3 min for 3-4 parameters and a partition of the channel cross section into 20 layers. An increase in the number of iterations (in x) to 50-100 does not alter the result in practice. An analogous problem, solved by the method and program in [1] uses 30-40 min on a BÉSM-6 computer and requires up to 1000 iterations. Both programs yields nearby results under the same conditions at the entrance and the number of nodes.

NOTATION

x, y, longitudinal and transverse coordinates; R, channel dimension; u, v, longitudinal and transverse velocity components; p, pressure; Π, parameter; ρ, density; μ, viscosity; Tu, turbulence level. Subscripts: i, I, number and total number of control nodes; H, initial (at the entrance); m, maximal (at the axis); 1, 2, 0, the sections $x = x_1$, $x = x_2$, $y = 0$, respectively; T, turbulent; and L, laminar.

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HEAT CONDUCTIVITY OF THE ADJOINING PLATES WITH A PLANE HEAT SOURCE BETWEEN THEM

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The heat conduction problem in a two-layered plate with a plane heat source between the layers is solved by the introduction of an unknown heat flux determined later from a Volterra integral equation of the second kind by the Bubnov–Galerkin method.

The Laplace transform method has limited application in solving heat conduction problems in two and multilayered walls because of the complexity of executing the inversion of the transform. A method is known for the reduction of such problems to the solution of Volterra type integral equations of the second kind in the unknown heat flux on the junction between the walls [1-3].

However, as is shown in [3], finite integral transforms result in a solution in the form of infinite poorly convergent series requiring the application of special methods to improve their convergence. Moreover, representation of the kernels of the Volterra integral equations in the form of infinite series does not permit obtaining the exact solution of the problem in analytic form.

It is expedient to use approximate methods based on the combined utilization of the Laplace integral transform and the Ritz or Bubnov–Galerkin method to solve heat conduction problems in multilayered walls. Such a method is developed in [4] for bodies of the simplest shape. In this case the solution of the Volterra integral equations relative to interlayered thermal fluxes, and therefore the solution of the problem is also successfully obtained in analytic form since the kernels of the integral equations consist of the simplest analytic functions without series.

It is shown in [4] that numerical values of the temperature fields obtained by using approximate and exact solutions agree with high accuracy.

The heat conduction problem considered in this paper is that a plane heat source, independent of the coordinates and time, acts between two infinite plates starting from a certain time. The heat transfer at the outer surfaces of the plates occurs according to the Newton law for a constant heat transfer coefficient. The thermophysical characteristics of the plates are independent of the temperature. The temperature of the plates at the

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